Individual Value of Oxford – Küstner.

In R.A.				In Decl.			
Value of Diff.	No. of Cases.	Value of Diff.	No. of Cases.	Value of Diff.	No. of Cases.	Value of Diff.	No. of Cases.
s		s		″		″	
−0·01	4	+0·16	25	−2·0	1	+0·1	22
0·00	1	+0·17	23	−1·9	2	+0·2	19
+0·01	2	+0·18	18	−1·3	1	+0·3	26
+0·02	3	+0·19	15	−1·2	5	+0·4	17
+0·03	3	+0·20	11	−1·0	2	+0·5	20
+0·04	8	+0·21	8	−0·9	9	+0·6	9
+0·05	6	+0·22	7	−0·8	8	+0·7	7
+0·06	7	+0·23	5	−0·7	19	+0·8	5
+0·07	15	+0·24	3	−0·6	15	+0·9	5
+0·08	19	+0·25	6	−0·5	20	+1·0	5
+0·09	16	+0·26	0	−0·4	29	+1·1	2
+0·10	25	+0·27	1	−0·3	20	+1·2	4
+0·11	17	+0·28	1	−0·2	35	+1·3	4
+0·12	24	+0·29	0	−0·1	24	+1·4	1
+0·13	34	+0·30	2	0·0	31		
+0·14	28	+0·31	1				
+0·15	28	+0·34	1				

General Conclusion.

Dr. Küstner's magnitude equation as determined by a comparison with Oxford photographic measures is in substantial agreement both in R.A. and Decl. with that found by him, using the method of screens; but the quantities are too small to settle the question whether the screen method is apt to give values which should be multiplied by a factor, as suggested in a former paper.

On Periodic Orbits in the Neighbourhood of Centres of Libration.

By H. C. Plummer, M.A.

1. It is well known that there are five exact solutions of the problem of three bodies in each of which the bodies preserve an unvarying configuration which revolves with a uniform velocity. It is also known that when the third body is of infinitesimal mass compared with the other two, it can describe small periodic orbits in the vicinity of the points where exact solutions exist. These points have been given by Gyldén the name of *centres of libration*, and a body describing an orbit of the kind indicated

has been called by Professor Darwin an *oscillating satellite.* Such orbits have been investigated recently by Dr. Charlier in a very interesting and suggestive paper.* The method which he has adopted is extremely simple and familiar, and several of his results had been given previously by Professor G. H. Darwin, in whose work the subject only presented itself incidentally, and by Mr. Moulton in his paper on the origin of the *Gegenschein.*† I wish here to discuss some of Dr. Charlier's results in a slightly more general manner. It is true that the direction in which greater generality can be secured without increased difficulty will lead us away from the problem of three bodies. But it is possible that the conclusions which follow will prove interesting in themselves, that they may conceivably find an application to some physical problem of a totally different order, and that something more may be learnt with regard to their general character and limitations. At any rate any attempt to draw wider attention to a subject which presents such a promising field for investigation may perhaps be justified.

2. Let us consider the motion of a particle in a rotating field of force. The forces are to be derived from a complete potential function Ω, the precise form of which need not at present be defined except as some uniform function of x and y, the coordinates referred to rotating axes. Then the equations of motion of the particle may by a suitable choice of units be expressed in the form

$$\ddot{x}-2\dot{y}=\frac{\partial\Omega}{\partial x}$$

$$\ddot{y}+2\dot{x}=\frac{\partial\Omega}{\partial y}$$

As is well known these apply in particular to the motion of a body of infinitesimal mass under the influence of two bodies revolving in circles about their centre of gravity. Now let a centre of libration be defined as a point whose coordinates (x, y) satisfy the relations

$$\frac{\partial\Omega}{\partial x}=\frac{\partial\Omega}{\partial y}=0$$

At such a point equilibrium, which may be either stable or unstable, is possible. Now let a surface be defined by the equation $z=\Omega$, and let the ordinary notation be adopted, viz.

$$\frac{\partial z}{\partial x}=p \qquad \frac{\partial z}{\partial y}=q$$

$$\frac{\partial^2 z}{\partial x^2}=r \qquad \frac{\partial^2 z}{\partial x\partial y}=s \qquad \frac{\partial^2 z}{\partial y^2}=t$$

* *Meddelanden från Lunds Astronomiska Observatorium*, No. 18.
† *Astr. Journ.*, No. 483.

Now if in the equations of motion we substitute $x+\xi$, $y+\eta$ for x and y and consider (x, y) a fixed point, we obtain

$$\ddot{\xi}-2\dot{\eta}=\frac{\partial\Omega}{\partial x}+\xi\frac{\partial^2\Omega}{\partial x^2}+\eta\frac{\partial^2\Omega}{\partial x\partial y}+\cdots$$

$$\ddot{\eta}+2\dot{\xi}=\frac{\partial\Omega}{\partial y}+\xi\frac{\partial^2\Omega}{\partial x\partial y}+\eta\frac{\partial^2\Omega}{\partial y^2}+\cdots$$

Hence if (x, y) is a centre of libration so that $p=q=0$ and if we neglect second and higher powers of ξ and η we may write

$$\ddot{\xi}-2\dot{\eta}=r\xi+s\eta$$

$$\ddot{\eta}+2\dot{\xi}=s\xi+t\eta$$

Then $\xi=\mathrm{A}e^{\lambda t}$, $\eta=\mathrm{B}e^{\lambda t}$ satisfy the equations provided

$$\mathrm{A}(\lambda^2-r)-\mathrm{B}(2\lambda+s)=0$$

$$\mathrm{A}(2\lambda-s)+\mathrm{B}(\lambda^2-t)=0$$

i.e. $$(\lambda^2-r)(\lambda^2-t)+4\lambda^2-s^2=0$$

or $$\lambda^4+\lambda^2(4-r-t)+rt-s^2=0$$

Stability depends on the nature of the roots of this equation.

3. Now let ρ_1, ρ_2 be the principal radii of curvature at the corresponding point of the surface $z=\Omega$. Since $p=q=0$ they are the roots of the quadratic

$$(rt-s^2)\rho^2+(r+t)\rho+1=0$$

Hence we may rewrite the preceding equation in λ^2 in the form

$$\lambda^4+\lambda^2\left(4+\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{1}{\rho_1\rho_2}=0$$

By means of this equation the conditions of stable equilibrium at a centre of libration are immediately connected with the properties of the surface $z=\Omega$. For perfect stability it is necessary that both values of λ^2 shall be real and negative. Hence we must in the first place have both ρ_1 and ρ_2 of the same sign, the surface synclastic at the point, and z a real maximum or a real minimum. The second condition is that

$$4+\frac{1}{\rho_1}+\frac{1}{\rho_2}>\frac{2}{+\sqrt{\rho_1\rho_2}}$$

If z is a maximum, ρ_1 and ρ_2 are positive, and the condition is necessarily fulfilled. If z is a minimum, ρ_1 and ρ_2 are negative, and the condition may be written

$$\frac{1}{+\sqrt{-\rho_1}}+\frac{1}{+\sqrt{-\rho_2}}<2$$

Thus, while in a stationary field of force equilibrium is stable where the force function is a maximum only, in a rotating field the function Ω, which it must be remembered includes the rotation potential, may be either a maximum or a minimum, provided in the latter case a certain inequality is satisfied. On the other hand we may have one value of λ^2 negative and the other positive. This corresponds to curvatures of opposite sign or an anticlastic point on the Ω surface. In this case a mode of periodic motion is combined with an unstable mode which must sooner or later prove fatal to the stable form. Neither purely positive nor imaginary values of λ^2 can be admitted in a search for periodic orbits.

4. I consider first the far more interesting case in which two periodic solutions are possible. Let us assume

$$\xi = h\cos(mt-\alpha) \qquad \eta = k\cos(mt-\beta)$$

as a periodic solution of the equations

$$\ddot{\xi} - 2\dot{\eta} = r\xi - s\eta$$
$$\ddot{\eta} + 2\dot{\xi} = s\xi + t\eta$$

The conditions obtained by substitution are

$$(r+m^2)\,h\cos\alpha + sk\cos\beta = -2mk\sin\beta$$
$$(r+m^2)\,h\sin\alpha + sk\sin\beta = \quad 2mk\cos\beta$$
$$(t+m^2)\,k\cos\beta + sh\cos\alpha = \quad 2mh\sin\alpha$$
$$(t+m^2)\,k\sin\beta + sh\sin\alpha = -2mh\cos\alpha$$

Hence we obtain directly

$$(r+m^2)\,h\sin(\alpha-\beta) = 2mk$$
$$sk\sin(\alpha-\beta) = -2mk\cos(\alpha-\beta)$$
$$(t+m^2)\,k\sin(\alpha-\beta) = \quad 2mh$$
$$sh\sin(\alpha-\beta) = -2mh\cos(\alpha-\beta)$$

which are equivalent to

$$h^2(r+m^2) = k^2(t+m^2)$$
$$s\tan(\alpha-\beta) = -2m$$
$$4m^2\operatorname{cosec}^2(\alpha-\beta) = (r+m^2)(t+m^2) = 4m^2+s^2$$

Hence the periodic orbit is obtained by combining two simple harmonic motions at right angles. These motions are characterised by a constant ratio of the amplitudes and a constant difference of phase for each centre of libration and assigned period. The orbits may be illustrated by Lissajou's figures for tuning-forks in unison. This is a very simple way of stating the

connection between the integration constants. The true arbitraries are, of course, the epoch and the scale of the orbit. Except for these the character of the orbits is absolutely determined by the nature of the field of force.

5. The equation of the orbit can be written down at once in the form

$$\frac{\xi^2}{h^2}+\frac{\eta^2}{k^2}-\frac{2\xi\eta}{hk}\cos(\alpha-\beta)=\sin^2(\alpha-\beta)$$

It will be found that an important distinction must be drawn in what follows, according as equilibrium corresponds to a maximum or minimum value of Ω. The latter case may be considered first. Then both r and t are positive, and we may put

$$c^2=h^2(r+m^2)=k^2(t+m^2)$$

and since

$$s(r+m^2)hk\sin^2(\alpha-\beta)=-4m^2k^2\cos(\alpha-\beta)$$

we have also

$$-hk\,.\,s\sec(\alpha-\beta)=c^2$$

The orbit becomes

$$(r+m^2)\xi^2+(t+m^2)\eta^2+2s\xi\eta=4m^2c^2/(4m^2+s^2)$$

and the axes and foci are given by

$$\frac{\xi^2-\eta^2}{r-t}=\frac{\xi\eta}{s}=-\frac{c^2}{4m^2+s^2}$$

The directions of the axes depend, then, on the form of the potential alone, and the major axis lies in the first and third quadrants if s is negative, in the second and fourth if s is positive. But for the surface $z=\Omega$

$$z+\delta z=\Omega+p\xi+q\eta+\tfrac{1}{2}(r\xi^2+2s\xi\eta+t\eta^2)+\ldots$$

or

$$2\delta z=r\xi^2+2s\xi\eta+t\eta^2$$

is the indicatrix corresponding to a centre of libration. The axes and foci are given by

$$\frac{\xi^2-\eta^2}{r-t}=\frac{\xi\eta}{s}=\frac{(2\delta z)^2}{s^2-rt}$$

and $s^2-rt=-\rho_1^{-1}\rho_2^{-1}$ is negative. Hence the major axes of the orbit and of the indicatrix are parallel. An alternative form of statement is that the axes of the orbit are parallel to the tangents to the lines of curvature at the corresponding point of the Ω surface.

6. The semi-axes a, b of the orbit are given by

$$\frac{1}{a^2}+\frac{1}{b^2}=(r+t+2m^2)\ (4m^2+s^2)\ /4m^2c^2$$

$$\frac{1}{a^2b^2}=(4m^2+s^2)^2/4m^2c^4$$

Hence a^2 and b^2 are the roots of the quadratic

$$(4m^2+s^2)^2\ x^2-(r+t+2m^2)\ (4m^2+s^2)c^2x+4m^2c^4=0$$

$$\begin{aligned}\therefore\ 2(4m^2+s^2)\ .\ x/c^2&=r+t+2m^2\\ &\qquad\pm\sqrt{\{4m^4+4m^2(r+t-4)+(r+t)^2\}}\\ &=r+t+2m^2\pm\sqrt{\{4(s^2-rt)+(r+t)^2\}}\\ &=2m^2-\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\pm\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\end{aligned}$$

We may therefore write

$$a^2=c^2\left(m^2-\frac{1}{\rho_1}\right)/(4m^2+s^2)$$

$$b^2=c^2\left(m^2-\frac{1}{\rho_2}\right)/(4m^2+s^2)$$

where ρ_1 is the numerically smaller of the two radii of curvature, for these are both negative. Accordingly the eccentricity e is given by

$$e^2=(\rho_2^{-1}-\rho_1^{-1})/(m^2-\rho_1^{-1})$$

and we notice that a circular orbit corresponds to an umbilic on the Ω surface.

7. We can now examine the initial conditions and the relation of our constant c to the constant C of the Jacobian integral

$$V^2=2\Omega-C$$

Let $(\xi_0.\ \eta_0)$ be the position at the time $t=0$. Then since

$$\xi=h\cos(mt-\alpha)\ \eta=k\cos(mt-\beta)$$

we have

$$\xi_0=h\cos\alpha\qquad\eta_0=k\cos\beta$$

$$\dot\xi_0=mh\sin\alpha\qquad\dot\eta_0=mk\sin\beta$$

$$\therefore\ V^2=\dot\xi_0^2+\dot\eta_0^2=m^2(h^2+k^2)-m^2(\xi_0^2+\eta_0^2)$$

But

$$2\Omega=2\Omega_0+r\xi_0^2+t\eta_0^2+2s\xi_0\eta_0$$

where Ω_0 is the value of Ω corresponding to the centre of libration. If $2\ \Omega_0=C_0$

$$C=C_0-m^2(h^2+k^2)+(r+m^2)\xi_0^2+(t+m^2)\eta_0^2+2s\xi_0\eta_0$$

Thus for a given value of $C-C_0$, ξ_0 and η_0 must satisfy the equation of a certain ellipse which can be no other than the orbit already found. Hence by comparison we find

$$\begin{aligned}C-C_0&=4m^2c^2/(4m^2+s^2)-m^2(h^2+k^2)\\&=\frac{4m^2c^2}{4m^2+s^2}-\frac{m^2c^2(2m^2+r+t)}{(m^2+r)(m^2+t)}\\&=m^2c^2(4-r-t-2m^2)/(4m^2+s^2)\end{aligned}$$

Now if m_1^2 and m_2^2 are the roots of the quadratic giving m^2

$$m_1^2+m_2^2=4-r-t$$

or

$$(2m_1^2+r+t-4)+(2m_2^2+r+t-4)=0$$

Hence $C-C_0$ clearly has opposite signs according to the period selected, that is, one period and one eccentricity correspond to an increase, and the other period and eccentricity to a decrease of the energy constant as compared with the value required for relative rest. Thus when stable equilibrium exists at a point where Ω is a minimum a given value of C determines the orbit uniquely. Dr. Charlier has noticed this in a particular case, but his proof is not of a general character. Another peculiarity is that one family of orbits is bounded by a set of "limiting curves," while the other is not.

8. The period of the orbit is $2\pi/m$. But the units have been so chosen that the period of rotation of the field of force is 2π. Therefore m may be defined as the ratio of the period of the rotating field to the synodic period of the orbit. The direction of motion is determined by the sign of

$$\begin{aligned}\xi\dot{\eta}-\dot{\xi}\eta&=-mhk\sin(\alpha-\beta)\\&=-2m^2c^2/(4m^2+s^2)=-mab\end{aligned}$$

The angular momentum being negative, the orbits of both families are retrograde. If it is desired to write down the equation of an orbit in a form involving only the energy constant and quantities depending only on Ω, we are led to

$$\begin{aligned}(r+m^2)\xi^2+(t+m^2)\eta^2+2s\xi\eta&=4(C-C_0)/(4-r-t-2m^2)\\&=4n(C-C_0)\end{aligned}$$

where

$$\begin{aligned}n^{-2}&=(4-r-t)^2-4(rt-s^2)\\&=\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)^2+8\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+16\end{aligned}$$

It may be observed that it is the family of shorter period and smaller eccentricity which requires $C<C_0$.

9. Throughout §§ 5–8 it has been assumed that Ω is an absolute minimum at the centre of libration, and that accordingly

r and t are positive. This indeed happens to be the more interesting case in so far as the problem of three bodies is concerned because there the function Ω does not as a matter of fact possess a truly maximum value. But if Ω is an absolute maximum both r and t are negative. It will be convenient to denote the quadratic giving m^2 by $f(m^2)=0$. By substitution we obtain:

$$f(+\infty) \text{ is positive.}$$
$$\left.\begin{array}{l} f(-r)=4r-s^2 \\ f(-t)=4t-s^2 \end{array}\right\} \text{ are negative.}$$
$$f(0) \;\; = rt - s^2 \text{ is positive.}$$

Hence $-r$ and $-t$ separate the two positive roots m_1^2 and m_2^2. It is clear then that when Ω is a maximum m_1^2+r and m_1^2+t are negative, while m_2^2+r and m_2^2+t are positive if $m_2^2 > m_1^2$. Further, since

$$4-r-t-2m_1^2 > 0$$

we must have

$$4-r-t-2m_2^2 < 0$$

The orbits of shorter period, $2\pi/m_2$, are then similar to the orbits investigated in the case when Ω is a minimum, and they require that $C < C_0$. It is unnecessary to consider this family in further detail. It is only in the expressions for the semi-axes and eccentricity given in § 6 that it is necessary to interchange the two radii of curvature on account of the change of sign.

10. The family of orbits characterised by the longer period $2\pi/m_1$ possesses features, however, which require separate notice. In the place of $+c^2$ in the foregoing it is necessary to write $-c^2$. The formula given for the foci in § 5 shows that the major axis of a member of this family is at right angles to the corresponding axis of a short-period orbit. Further from § 8 it appears that the direction of motion is reversed and orbits of this family are direct. Since both c^2 and $(4-r-t-2m^2)$ have opposite signs in the two families we again have $C < C_0$. Consequently to any given value of $C < C_0$ one orbit of each family corresponds. Indeed the Jacobian integral suffices to prove that $C \not> C_0$; for otherwise since $2\Omega_0 > 2\Omega$, the velocity of the displaced body would be imaginary. Hence the extension of Dr. Charlier's statement given in § 7 of this paper, must be restricted to cases where Ω is a minimum. The expressions given for the semi-axes in § 6 require no alteration, although the radii of curvature are positive: for $(m_1^2-\rho_1^{-1})$ and $(m_1^2-\rho_2^{-1})$ must be negative, and therefore $\rho_1^{-1} > \rho_2^{-1}$. The most interesting features of the family of orbits considered here are undoubtedly the change from retrograde to direct motion and the rotation of the major axis through a right angle.

11. When Ω is a maximum at the centre of libration, we have seen that two orbits correspond to one value of C. Again,

through a point two orbits can be drawn, and for these the values of C will in general be different. There are, however, points where two orbits of different families, and the same C meet. The orbits will be

$$(m_1^2+r)\xi^2+(m_1^2+t)\eta^2+2s\xi\eta=+4n(C-C_0)$$
$$(m_2^2+r)\xi^2+(m_2^2+t)\eta^2+2s\xi\eta=-4n(C-C_0)$$

Hence the locus of their intersection is

$$(m_1^2+m_2^2+2r)\xi^2+(m_1^2+m_2^2+2t)\eta^2+4s\xi\eta=0$$

Or since
$$m_1^2+m_2^2=4-r-t$$
$$(4+r-t)\xi^2+(4-r+t)\eta^2+4s\xi\eta=0$$

This represents a pair of straight lines equally inclined to the axes of the orbit, and this is clear from the geometrical interpretation. For since C is the same for both orbits, the velocity in each at the point of intersection is the same. Hence the central perpendiculars on the tangents are equal, and it becomes evident that the above equation represents the locus of points where orbits of the two families intersect each other, and are equally inclined to the common radius vector. The locus will not be real unless

$$4s^2+(r-t)^2>16$$

i.e.
$$\rho_1^{-1}\sim\rho_2^{-1}>4$$

And unless this condition is satisfied corresponding members of the two families will not intersect, so that one orbit will lie wholly within the other.

12. When at a centre of libration the function Ω is neither an absolute maximum nor an absolute minimum, oscillating satellites may still exist. In this case only one class of orbit is possible, and even this cannot be permanently stable. The equation $f(m^2)=0$ has one positive root and one negative. Since the equation

$$(2m_1^2+r+t-4)+(2m_2^2+r+t-4)=0$$

still holds, the first expression on the left hand must be positive if $m_1^2>0>m_2^2$. Hence $(2m_1^2+r+t)$ is positive, and since (m_1^2+r) and (m_1^2+t) must be of the same sign, if an elliptic orbit exists they must both be positive. Hence $C<C_0$, and the orbits possess the same general features as those of shorter period when Ω is a minimum. An unstable form of motion arises immediately if $C>C_0$, and this would be the case if motion in the periodic orbit be unduly retarded. We should infer that excess rather than defect of relative velocity would be favourable to the duration of quasi-stable motion, and this may have some bearing on the theory of the detention of meteorites in the neighbourhood of a centre of libration. We have seen that in general the direction

of the major axis of an orbit depends on the sign of s. An exception, of course, arises when $s=0$. This point happens to occur in the application of this paragraph, and may properly be considered in this connection. The formula given in § 5 for the foci of the orbit shows that the axes are parallel to the rotating axes of coordinates, and that the major axis is parallel or perpendicular to the line joining the controlling bodies, according as $r-t <$ or > 0. The only exception to this rule is in the case of long-period orbits corresponding to Ω a maximum, when the preceding inequalities are reversed.

13. About the special problem which suggested the foregoing slightly more general investigation, little need be said here, because it has been treated with some fullness by Dr. Charlier. But as I have had occasion to allude to the properties of the surface defined by the equation $z=\Omega$, it may not be out of place to consider briefly the nature of the surface determined by two bodies revolving in circles about their centre of gravity. The sum of the masses μ and ν is taken as the unit of mass, while the unit of time is such that the gravitational constant is unity and the unit of length the distance between the masses. Then the period $T=2\pi$, the angular velocity is unity, and we have

$$2\Omega=\mu\left(r_1^2+\frac{2}{r_1}\right)+\nu\left(r_2^2+\frac{2}{r_2}\right)$$

where r_1, r_2 are the distances from μ and ν respectively. Now Professor Darwin in his memoir * has drawn a typical series of the curves $2\ \Omega=$const., and if these be regarded not merely as a family of curves with an arbitrary parameter, but as a set of contour lines of a surface, we easily get a fair notion of the general character of that surface. Much indeed can be gathered from the simplest reasoning. At the two points where $r_1=r_2=1$, it is easy to prove that $2\ \Omega$ is a true minimum and equal to 3. The familiar topographical illustration will be useful here. At each of these minimum points there is a valley. The function Ω becomes infinite at the points where the masses are placed, but for the present purpose these points may be represented by the summits of mountains. The contour lines, which are evidently symmetrical with respect to the line joining the masses, must rapidly approach a circular form as they recede from the centres of force and indicate rising ground in all directions outwards. The additional features of the surface must clearly be three passes, one running between the mountains and connecting the valleys, and two others beyond the two mountains and separating them from the ground rising in the distance. Each of the three passes has its highest point on the line of symmetry, and these three points correspond to the three collinear centres of libration. Here the Ω surface is anticlastic, and, as indeed is evident on dynamical grounds, stable equilibrium is out of the question.

* *Acta Math.*, t. xxi. p. 113.

14. Although there cannot be any doubt as to the nature of these points, yet it may be permissible to consider them from another point of view. In what follows the origin is taken at the centre of gravity of the revolving bodies, the latter always lying on the rotating axis of x. Since

$$2\Omega = \mu\left(r_1^2 + \frac{2}{r_1}\right) + \nu\left(r_2^2 + \frac{2}{r_2}\right)$$

we have

$$\frac{\partial\Omega}{\partial r_1} = \mu\left(r_1 - \frac{1}{r_1^2}\right) ; \frac{\partial\Omega}{\partial r_2} = \nu\left(r_2 - \frac{1}{r_2^2}\right)$$

Also for the collinear centres of libration

$$y = 0 ; \frac{\partial\Omega}{\partial x} = 0$$

or

$$\frac{\partial\Omega}{\partial r_1} \cdot \frac{\partial r_1}{\partial x} + \frac{\partial\Omega}{\partial r_2} \cdot \frac{\partial r_2}{\partial x} = 0$$

Moreover
$$\frac{\partial\Omega}{\partial y} = \frac{\partial\Omega}{\partial r_1} \cdot \frac{\partial r_1}{\partial y} + \frac{\partial\Omega}{\partial r_2} \cdot \frac{\partial r_2}{\partial y}$$
$$= \frac{y}{r_1} \cdot \frac{\partial\Omega}{\partial r_1} + \frac{y}{r_2} \cdot \frac{\partial\Omega}{\partial r_2}$$

Hence when $y = 0$

$$t = \frac{\partial^2\Omega}{\partial y^2} = \frac{1}{r_1} \cdot \frac{\partial\Omega}{\partial r_1} + \frac{1}{r_2} \frac{\partial\Omega}{\partial r_2}$$

$$s = \frac{\partial^2\Omega}{\partial x \partial y} = 0$$

We have now to treat separately the cases which arise.

(1) Between the masses μ and ν we have

$$r_1 + r_2 = 1 ; \frac{\partial r_1}{\partial x} + \frac{\partial r_2}{\partial x} = 0$$

$$\therefore \frac{\partial\Omega}{\partial r_1} = \frac{\partial\Omega}{\partial r_2}$$

$$\therefore \mu\left(\frac{1}{r_1^2} - r_1\right) = \nu\left(\frac{1}{r_2^2} - r_2\right)$$

Hence clearly $r_1 > r_2$ if $\mu > \nu$. In this case

$$t = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot \mu\left(r_1 - \frac{1}{r_1^2}\right)$$

and is negative, for $r_1 < 1$.

(2) Beyond the mass μ we have

$$r_2 = r_1 + 1\ ;\ \frac{\partial r_1}{\partial x} = \frac{\partial r_2}{\partial x}$$

$$\therefore\ \frac{\partial \Omega}{\partial r_1} + \frac{\partial \Omega}{\partial r_2} = 0$$

$$\therefore\ \mu\left(\frac{1}{r_1^2} - r_1\right) = \nu\left(r_2 - \frac{1}{r_2^2}\right)$$

Hence clearly $r_1 < 1$. In this case

$$t = \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \cdot \mu\left(r_1 - \frac{1}{r_1^2}\right)$$

and is negative because $r_1 < 1 < r_2$, and therefore the first factor is positive and the last negative. The proof that t is negative in the case of the centre of libration lying beyond the mass ν can be inferred from the latter, as there is no restriction on the ratio of the masses μ and ν. Now we have

$$r = \frac{\partial^2 \Omega}{\partial x^2} = \mu\left(1 + \frac{2}{r_1^3}\right) + \nu\left(1 + \frac{2}{r_2^3}\right)$$

which is necessarily positive. Hence always for the three points considered

$$rt - s^2 < 0$$

and therefore the two roots of the quadratic in m^2 are of opposite signs. I have thought it worth while to give a definite proof of this fact because on this point Dr. Charlier says : "That is what I have found through numerical calculation for different values of μ, an algebraic demonstration of this fact seeming to be somewhat complicated."

University Observatory, Oxford :
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A New Method of Interpolation. By T. C. Hudson, B.A.

1. Bessel's interpolation formula is

$$u_n - u_0 = n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{1\,.\,2}\,\frac{\Delta_0'' + \Delta_1''}{2} + \frac{n(n-1)(n-\frac{1}{2})}{1\,.\,2\,.\,3}\,\Delta_{\frac{1}{2}}''' + \frac{(n+1)n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4}\,\frac{\Delta_0^{\mathrm{iv}} + \Delta_1^{\mathrm{iv}}}{2} + \ \ldots \qquad (1)$$

2. As it stands this formula is partially symmetrical with respect to the middle of the interval ; but there is no symmetry